

Probabilistic Logic: Many-valuedness and Intensionality

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Abstract

The probability theory is a well-studied branch of mathematics, in order to carry out formal reasoning about probability. Thus, it is important to have a logic, both for computation of probabilities and for reasoning about probabilities, with a well-defined syntax and semantics. Both current approaches, based on Nilsson's probability structures/logics, and on linear inequalities in order to reason about probabilities, have some weak points.

In this paper we have presented the complete revision of both approaches. We have shown that the full embedding of Nilsson's probabilistic structure into propositional logic results in a truth-functional many-valued logic, differently from Nilsson's intuition and current considerations about propositional probabilistic logic. Then we have shown that the logic for reasoning about probabilities can be naturally embedded into a 2-valued intensional FOL with intensional abstraction, by avoiding current ad-hoc system composed of two different 2-valued logics: one for the classical propositional logic at lower-level, and a new one at higher-level for probabilistic constraints with probabilistic variables. The obtained theoretical results are applied to Probabilistic Logic Programming.

1 Introduction

The main motivation for an introduction of the intensionality in the probabilistic-theory of the propositional logic is based on the desire to have the *full* logical embedding of the probability into the First-Order Logic (FOL), with a clear difference from the classic concept of truth of the logic formulae and the concept of their probabilities. In this way we are able to replace the ad-hoc syntax and semantics, used in current practice for Probabilistic Logic Programs [1, 2, 3, 4] and probabilistic deduction [5], by the standard syntax and seman-

tics used for the FOL where the probabilistic-theory properties are expressed simply by the particular constraints on their interpretations and models.

In this paper we will consider the probabilistic semantics for the propositional logic (it can be easily extended to predicate logics as well) [6, 7] with a fixed finite set $\Phi = \{p_1, \dots, p_n\}$ of primitive propositions, which can be thought of as corresponding to basic probabilistic events. The set $L(\Phi)$ of the propositional formulae is the closure of Φ under the Boolean operations for conjunction and negation, \wedge and \neg , that is, it is the set of all formulae of the propositional logic $(\Phi, \{\wedge, \neg\})$. In order to give the probabilistic semantics to such formulae, we first need to review briefly the probability theory (see, for example, [8, 9]):

Definition 1 *A probability space (S, \mathcal{X}, μ) consists of a set S , called the sample space, a σ -algebra \mathcal{X} of subsets of S (i.e., a set of subsets of S containing S and closed under complementation and countable union, but not necessarily consisting of all subsets of S) whose elements are called measurable sets, and a probability measure $\mu : \mathcal{X} \rightarrow [0, 1]$ where $[0, 1]$ is the closed interval of reals from 0 to 1. This mapping satisfies Kolmogorov axioms [10]:*

A.1 $\mu \geq 0$ for all $X \in \mathcal{X}$.

A.2 $\mu(S) = 1$.

A.3 $\mu(\bigcup_{i \geq 1} X_i) = \sum_{i \geq 1} \mu(X_i)$, if X_i 's are nonempty pairwise disjoint members of \mathcal{X} .

The $\mu(\{s\})$ is the value of probability in a single point of space s .

The property A.3 is called *countable additivity* for the probabilities in a space S . In the case when \mathcal{X} is finite set, then we can simplify property A.3 above to

A.3' $\mu(X \cup Y) = \mu(X) + \mu(Y)$,

if X and Y are disjoint members of \mathcal{X} , or, equivalently, to the following axiom:

A.3'' $\mu(X) = \mu(X \cap Y) + \mu(X \cap \bar{Y})$,

where \bar{Y} is the compliment of Y in S , so that $\mu(\bar{X}) = 1 - \mu(X)$.

In what follows we will consider only finite sample space S , so that $\mathcal{X} = \mathcal{P}(S)$ is the set of all subsets of S . Thus, in our case of a finite set S we obtain, from A.1 and A.2, that for any $X \in \mathcal{P}(S)$, $\mu(X) = \sum_{s \in X} \mu(\{s\})$.

Based on the work of Nilsson in [6] we can define for a given propositional logic with a finite set of primitive proposition Φ the sample space $S = 2^\Phi$, where $2 = \{0, 1\} \subset [0, 1]$, so that the probability space is equal to the Nilsson structure $N = (2^\Phi, \mathcal{P}(2^\Phi), \mu)$.

In his work (page 2, line 4-6 in [6]) Nilsson considered a Probabilistic Logic "in which the truth values of sentences can range between 0 and 1. The truth value of a sentence in *probabilistic logic* is taken to be the probability of that sentence in ordinary first-order logic." That is, he considered this logic as a kind of a *many-valued*, but not a compositional truth-valued, logic. But in his paper he did not defined the formal syntax and semantics for such a probabilistic logic, but only the matrix equations where the probability of a sentence $\phi \in \mathcal{L}(\Phi)$ is the sum of the probabilities of the sets of possible worlds (equal to the set $S = 2^\Phi$) in which that sentence is *true*. So that he assigns *two* different logic values to each sentence ϕ : one is its probability value and another is a classic 2-valued truth value in a given possible world. It is formally contradictory with his intension paraphrased above. In fact, as we will see in one of the following Section, the correct formalization of such a many-valued logic *with probabilistic semantics* is different, and more complex, from his intuitive initial idea.

The *logic* inadequacy of this seminal work [6] of Nilsson is also considered in [11], by extending this Nilsson structure $N = (2^\Phi, \mathcal{P}(2^\Phi), \mu)$ into a more general *probability structure* $M = (2^\Phi, \mathcal{P}(2^\Phi), \mu, \pi)$, where π associates with each $s \in S = 2^\Phi$ the truth assignment $\pi(s) : \Phi \rightarrow 2$ in the way that we say that $p \in \Phi$ is *true at s* if $\pi(s)(p) = 1$; *false at s* otherwise. This mapping $\pi(s)$ can be uniquely extended to the truth assignment on all formulae in $\mathcal{L}(\Phi)$, by taking the usual rules of propositional logic (the unique homomorphic extension to all formulae), and we can associate to each propositional formula $\phi \in \mathcal{L}(\Phi)$ the set ϕ^M consisting of all states $s \in S$ where ϕ is true (so that $\phi^M = \{s \in S \mid \pi(s)(\phi) = 1\}$). In [11] it was demonstrated that for each Nilsson structure N there is an equivalent measurable probability structure M , and vice versa.

But, differently from Nilsson, they did not define a many-valued propositional logic, but a kind of 2-valued logic based on probabilistic constraints. They denoted by $w_N(\phi)$ the *weight* or *probability* of ϕ in Nilsson structure N , correspondent to the value $\mu(\phi^M)$, so that the

basic probabilistic 2-valued constraint can be defined by expressions $c_1 \leq w_N(\phi)$ and $w_N(\phi) \leq c_2$ for given constants $c_1, c_2 \in [0, 1]$. They expected their logic to be used for reasoning *about* probabilities.

But, again, from the logic point of view, they did not defined a unique logic, but *two different* logics: one for the classical propositional logic $\mathcal{L}(\phi)$, and a new one for 2-valued probabilistic constraints obtained from the basic probabilistic formulae above and Boolean operators \wedge and \neg . They did not consider the introduced symbol w_N as a formal functional symbol for a mapping $w_N : \mathcal{L}(\Phi) \rightarrow [0, 1]$, such that for any propositional formula $\phi \in \mathcal{L}(\Phi)$, $w_N(\phi) = \mu(\{s \in S \mid \pi(s)(\phi) = 1\})$. Instead of this intuitive meaning for w_N they considered each expression $w_N(\phi)$ as a particular probabilistic term (more precisely, as a structured probabilistic *variable* over the domain of values in $[0, 1]$).

It seems that such a dichotomy and difficulty to have a *unique* 2-valued probabilistic logic, both for an original propositional formulae in $\mathcal{L}(\Phi)$ and for the probabilistic constraints, is based on the fact that if we consider w_N as a function with one argument then it has to be formally represented as a binary predicate $w_N(\phi, a)$ (for the graph of this function) where the first argument is a formula and the second is its resulting probability value. Consequently, a constraint "the probability of ϕ is less or equal to c ", has to be formally expressed by the logic formula $w_N(\phi, a) \wedge \leq (a, c)$ (here we use a symbol \leq as a built-in rigid binary predicate where $\leq (a, c)$ is equivalent to $a \leq c$), which is a *second-order* syntax because ϕ is a logic *formula* in such a unified logic language. That is, the problem of obtaining the unique logical framework for probabilistic logic comes out with the necessity of a *reification* feature of this logic language, analogously to the case of the intensional semantics for RDF data structures [12].

Consequently, we need a logic which is able to deal directly with reification of logic formulae, and this is the starting point of this work. In fact as we will see, such an unified logical framework for the probabilistic theory can be done by a kind of predicate intensional logics with intensional abstracts, that transforms a propositional formulae $\phi \in \mathcal{L}(\Phi)$ into an abstracted *term*, denoted by $\langle\phi\rangle$. By this approach the expression $w_N(\langle\phi\rangle, a) \wedge \leq (a, c)$ remains to be an ordinary first-order formula. In fact, if $\langle\phi\rangle$ is translated into non-sentence "that ϕ ", then the first-order formula above corresponds to the sentence "the probability **that** ϕ is true is less than or equal to c ".

More about the intensional FOL can be found in the Appendix. The decision to define the intensional FOL with abstraction in Appendix, instead in the first Section is based only on the fact that main issue of this pa-

per is the probabilistic logic and not intensional logic. But for readers without previous experience about the intensional FOL, we recommend to read this Appendix before the Section 3. Also the particular development of this intensional FOL is an original contribution of this paper. The Plan of this work is the following:

In Section 2 we investigate the relationship between Nilsson's probability structure and many-valued propositional logic. We define an algebraic probabilistic propositional logic for Nilsson's structure and show that it is correct truth-functional many-valued logic for computation of probabilities analogously to Nilsson's structure.

In Section 3 we define an embedding of the probability theory, both with reasoning about probabilities, into an intensional FOL with intensional abstraction. Then we show that the probabilities of propositional formulae corresponds to the computation of their probabilities in Nilsson's structures, that is, this intensional FOL is sound and complete w.r.t the measurable probability structures.

Finally, in Section 4 we apply the theoretical results, obtained in previous two Sections, to Probabilistic Logic Programming.

2 Probabilistic logic and many-valuedness

The work of Jan Lukasiewicz was without doubt the most influential in the development of many-valued modal logics [13, 14, 15, 16]. In Lukasiewicz's conception, the real definition of a logic must be semantic and truth-functional (the logic connectives are to be truth functions operating on these logical values): "logic is the science of objects of a specific kind, namely a science of *logical values*". Many-valued logics are non-classical logics. They are similar to classical logic because they accept the principle of truth-functionality, namely, that the truth of a compound sentence is determined by the truth values of its component sentences (and so remains unaffected when one of its component sentences is replaced by another sentence with the same truth value). But they differ from classical logic by the fundamental fact that they do not restrict the number of truth values to only two: they allow for a larger set of truth degrees.

The Lukasiewicz's work promoted the concept of *logic matrix*, central concept for the construction of many-valued logics, implicit in the works of C.S.Pierce and E.Post as well. We will briefly present the previous work for many-valued propositional logics, based on such matrices: The representation theorems [17] for such logics are based on Lindenbaum algebra of a logic

$L = (\Phi, \emptyset, \Vdash)$, where Φ is a set of propositional variables of a language L , \emptyset is the set of logical connectives and \Vdash is the entailment relation of this logic.

We denote by $L(\Phi)$ the set of all formulae. Lindenbaum algebra of L is the quotient algebra $L(\Phi)/\equiv$, where for any two formulae $\phi, \psi \in L(\Phi)$, holds the equivalence $\phi \equiv \psi$ iff $\phi \Vdash \psi$ and $\psi \Vdash \phi$. The standard approach to matrices uses a subset $D \subset A$ of the set of truth values A (nullary operators, i.e., logic constants), denominated designated elements; informally a designated element represent an equivalence class of the *theorems* in L . Given an algebra $\mathbf{A} = (A, \{o\}_{o \in \emptyset})$, the \emptyset -matrix is the pair (\mathbf{A}, D) , where $D \subset A$ is a subset of designated elements. The *algebraic* satisfaction relation \models^a ('a' stands for 'algebraic') is defined as follows:

Definition 2 Let $L = (\Phi, \emptyset, \Vdash)$ be a logic, (\mathbf{A}, D) a \emptyset -matrix, and $\phi \in L(\Phi)$. Let $v : \Phi \rightarrow A$ be a mapping that assigns the logic values to propositional variables, and $\bar{v} : L(\Phi) \rightarrow A$ be its unique extension to all formulae of this logic language $L(\Phi)$. We define the relation \models^a inductively as follows:

1. $(\mathbf{A}, D) \models_v^a \phi$ iff $\bar{v}(\phi) \in D$,
2. $(\mathbf{A}, D) \models^a \phi$ iff $\bar{v}(\phi) \in D$ for every $v : \Phi \rightarrow A$.

We say that a valuation \bar{v} is a *model* for a subset of propositional formulae $\Gamma \subset L(\Phi)$ if for all formulae $\phi \in \Gamma$ it holds that $(\mathbf{A}, D) \models_v^a \phi$.

Any modal 2-valued logic can be considered as many-valued truth-functional logic for a given set \mathcal{W} of possible worlds as well, given by a complex algebra $\mathbf{A} = (A, \{\cap, \cup, /, m, l\})$ of "truth-values", where $A = \mathcal{P}(\mathcal{W})$ is the powerset of complete lattice of "truth-values" (empty set \emptyset is the bottom, while \mathcal{W} is the top "truth-value") and $\cap, \cup, /$ are algebraic operations for conjunction, disjunction and negation (that is, the set-intersection, set-union and set-complement operators) respectively, while m, l are algebraic operations for universal and existential modal logic operators: in this approach the "truth-value" of a given sentence ϕ is equal to the subset of possible worlds $\|\phi\|$ where this formula ϕ is satisfied, i.e., equal to $\{w \in \mathcal{W} \mid \mathcal{M} \models_w \phi\} \in \mathcal{P}(\mathcal{W})$. In such an algebraic logic of a modal logic, the \emptyset -matrix is a pair (\mathbf{A}, D) where the set of designated values is a singleton $D = \{\mathcal{W}\}$. Thus, a sentence ϕ is "true in a Kripke model $\mathcal{M} = (\mathcal{W}, \mathcal{F}, v)$ " with a frame \mathcal{F} of this modal logic iff $(\mathbf{A}, D) \models_v^a \phi$, that is, iff $\bar{v}(\phi) \in D$ (i.e., iff $\bar{v}(\phi) = \mathcal{W}$), that is, if the "truth-value" of ϕ is the top value in $A = \mathcal{P}(\mathcal{W})$. Notice that instead of this set-based algebra $\mathbf{A} = (A, \{\cap, \cup, /, m, l\})$ we can use the functional algebra $\mathbf{A} = (\bar{A}, \{\bar{\cap}, \bar{\cup}, \bar{/}, \bar{m}, \bar{l}\})$ with higher order "truth values" [18, 19] given by functions in the

functional space $\tilde{A} = \mathbf{2}^{\mathcal{W}}$, where for each $S \in \mathcal{P}(\mathcal{W})$ we have the correspondent functional truth-value $f \in \mathbf{2}^{\mathcal{W}}$, such that for any $w \in \mathcal{W}$ it holds that $f(w) = 1$ iff $w \in S$. For example, the function $f \tilde{\cap} g : \mathcal{W} \rightarrow \mathbf{2}$ is defined for any $s \in \mathcal{W}$ by $(f \tilde{\cap} g)(s) = f(s) \cdot g(s)$, while $(f \tilde{\cup} g)(s) = f(s) + g(s)$ and $(\tilde{\neg}(f))(s) = 1 - f(s)$. In what follows, for any function $f \in \mathbf{2}^S$ we denote its image by $Im(f) = \{s \in S \mid f(s) = 1\}$, so that $Im(f \tilde{\cap} g) = Im(f) \cap Im(g)$ and $im(\tilde{\neg}(f)) = S/Im(f)$. Here we consider the possibility to have also infinite matrices for many-valued truth-functional logics, differently from other approaches that consider only finite matrices [20]. In fact, based on Nilsson probabilistic structure $N = (\mathbf{2}^\Phi, \mathcal{P}(\mathbf{2}^\Phi), \mu)$, we can define the following Probabilistic algebra:

Definition 3 Let $N = (\mathbf{2}^\Phi, \mathcal{P}(\mathbf{2}^\Phi), \mu)$ be a Nilsson structure with a sample space $S = \mathbf{2}^\Phi$. Then we define the probabilistic algebra $\mathbf{A} = (\mathbf{2}^S \times [0, 1], \{\wedge, \sim\})$ with the \mathcal{O} -matrix (\mathbf{A}, D) where $D = \{(f, \mu(im(f))) \mid f \in \mathbf{2}^S\}$ is a set of designated elements, such that the binary operator "p-conjunction" \wedge and unary operator "p-negation" are defined as follows: for any two $(f, x), (g, y) \in A = \mathbf{2}^S \times [0, 1]$,

$$\begin{aligned} \wedge((f, x), (g, y)) &= (f \tilde{\cap} g, \mu(Im(f \tilde{\cap} g))) \\ &\quad \text{if } x = \mu(im(f)), y = \mu(im(g)); \\ &\quad (f \tilde{\cap} g, 0) \text{ otherwise,} \\ \sim(f, x) &= (\tilde{\neg}(f), \mu(\tilde{\neg}(f))) \text{ if } x = \mu(im(f)); \\ &\quad (\tilde{\neg}(f), 0) \text{ otherwise.} \end{aligned}$$

Notice that each truth-value of this algebra is a pair of elements $a \in A$: first one $\pi_1(a)$ defines the set of possible worlds in S where a propositional formula in $L(\Phi)$ is satisfied, while the second element $\pi_2(a)$ is the probability of this formula (here $\pi_i, i = 1, 2$ are first and second cartesian projections).

The set of truth-values in A is infinite, but its subset of designated elements is finite for the finite set of propositional variables in Φ .

Let us show that this algebra represents a *truth-functional* many-valued semantics for the propositional Nilsson's probabilistic logic.

Proposition 1 Let $N = (\mathbf{2}^\Phi, \mathcal{P}(\mathbf{2}^\Phi), \mu)$ be a Nilsson structure with a sample space $S = \mathbf{2}^\Phi$, with the \mathcal{O} -matrix (\mathbf{A}, D) given by Definition 3, and where $v : \Phi \rightarrow A$ is a mapping that assigns the logic values to propositional variables, such that for any $p \in \Phi$, $v(p) = (f, \mu(Im(f)))$ where for any $s \in S = \mathbf{2}^\Phi$ it holds that $f(s) = s(p)$. We denote by $\bar{v} : (\Phi, \{\wedge, \neg\}) \rightarrow \mathbf{A}$ the unique homomorphic extension of v to all formulae of the propositional logic $(\Phi, \{\wedge, \neg\})$.

Then, for any propositional formula $\phi \in L(\Phi)$ we have

that $(\mathbf{A}, D) \models_v^a \phi$ implies that $\pi_2(\bar{v}(\phi))$ is the Nilsson's probability of ϕ .

That is, a many-valued truth-functional assignment \bar{v} is a model for this propositional probabilistic logic.

Proof: Let us demonstrate this proposition by structural induction on formulae $\phi \in L(\Phi)$:

1. For any basic propositional formula $p \in \Phi$ we have that $(\mathbf{A}, D) \models_v^a p$ (it holds that $v(p) = (f, \mu(Im(f))) \in D$).

Let us suppose that $\phi_1, \phi_2 \in L(\Phi)$ satisfy this property, that is, $\bar{v}(\phi_1) = (f, \mu(Im(f))) \in D$ and $\bar{v}(\phi_2) = (g, \mu(Im(g))) \in D$. Then we have the following two cases:

2.1 Case when $\phi = \phi_1 \wedge \phi_2$. Then $\bar{v}(\phi) = \bar{v}(\phi_1 \wedge \phi_2) = \wedge(\bar{v}(\phi_1), \bar{v}(\phi_2))$ (from the homomorphic property) $= \wedge((f, \mu(Im(f))), (g, \mu(Im(g))))$

$= (f \tilde{\cap} g, \mu(Im(f \tilde{\cap} g)))$ (from Definition 3)

$= (f \tilde{\cap} g, \mu(Im(f \tilde{\cap} g))) \in D$ (from $f \tilde{\cap} g \in \mathbf{2}^S$).

2.2 Case when $\phi = \neg \phi_1$. Then $\bar{v}(\phi) = \bar{v}(\neg \phi_1) = \sim(\bar{v}(\phi_1))$ (from homomorphic property)

$= \sim(f, \mu(Im(f)))$

$= (\tilde{\neg}(f), \mu(S/Im(f)))$ (from Definition 3)

$= (\tilde{\neg}(f), \mu(Im(\tilde{\neg}(f)))) \in D$ (from $\tilde{\neg}(f) \in \mathbf{2}^S$).

Thus, for any $\phi \in L(\Phi)$ we have that $\bar{v}(\phi) = (f, \mu(Im(f)))$, where $f : S \rightarrow \mathbf{2}$ satisfies for any $s \in S = \mathbf{2}^\Phi$ that $f(s) = 1$ if $\bar{s}(\phi) = 1$, so that $\mu(Im(f)) = \pi_2(\bar{v}(\phi))$ is the Nilsson's probability of the formula ϕ .

From the fact that for any $\phi \in L(\Phi)$ holds that $(\mathbf{A}, D) \models_v^a \phi$ we deduce that \bar{v} is a model for this propositional probabilistic logic.

□

Notice that for a given propositional logic $(\Phi, \{\wedge, \neg\})$ and Nilsson's structure $N = (\mathbf{2}^\Phi, \mathcal{P}(\mathbf{2}^\Phi), \mu)$, a model $\bar{v} : (\Phi, \{\wedge, \neg\}) \rightarrow \mathbf{A}$ computes the probabilities of all formulae in $L(\Phi)$. But there is no way for this many-valued propositional logic $(\Phi, \{\wedge, \neg\})$ to reason about these probabilities.

Notice that this approach to probabilistic many-valued logic demonstrate that, differently from Nilsson's remark (1.3-5, p.72 in [6]):

- "...we present a semantic generalization of ordinary first-order logic in which the truth values of sentences can range between 0 and 1. The truth-value of a sentence in *probabilistic logic* is taken to be the *probability* of that sentence .."

the truth-value of a sentence is *not* the probability of that sentence but the pair (f, a) where the first element $f \in \mathbf{2}^S$ is a mapping from the sample space $S = \mathbf{2}^\Phi$ into the set $\mathbf{2}$; only its second component $a \in [0, 1]$ is the probability of that sentence. It demonstrates that

his intuition was only god but approximative one, so that we needed this complete and formal revision of his original intuition.

The second consequence is that, differently from current opinion in the computer science community that the probabilistic logic is not truth-functionally many-valued logic, we demonstrated that indeed *it is*: the truth-value of a complex sentence is functionally dependent on the truth-values of its proper subsentences, as in standard many-valued logics.

3 Probabilistic logic and intensionality

In order to reason about probabilities of the propositional formulae we need a kind of 2-valued meta-logic with *reification* features, thus, a kind of intensional FOL with *intensional abstraction* presented in the Appendix.

Intensional entities are such things as concepts, propositions and properties. The *intensional abstracts* are 'that'-clauses. For example, in the intensional sentence "it is necessary that ϕ ", where ϕ is a proposition, the 'that ϕ ' is denoted by the $\langle\phi\rangle$, where $\langle\phi\rangle$ is the intensional abstraction operator which transforms a logic formula into a *term*. So that the sentence "the probability that A is less then or equal to c" is expressed by the first-order logic formula $w_N(\langle\phi\rangle, a) \wedge \leq(a, c)$, where \leq is the binary built-in predicate 'is less then or equal', where the usual notation " $a \leq b$ " is rewritten in this standard predicate-based way by " $\leq(a, b)$ ", while "the probability that ϕ is equal to a " is denoted by the ground atom $w_N(\langle\phi\rangle, a)$ with the binary "functional" predicate symbol w_N (in intensional logic any n-ary function is represented by the $n+1$ -ary predicate symbol with first n attributes used as arguments of this function and last $(n+1)$ -th attribute for the resulting function's value, analogously as in FOL with identity). The intensional logic thus is a FOL with terms of FOL and terms obtained by applying intensional abstraction to the formulae. We consider a non empty domain $\mathcal{D} = D_{-1} \cup D_I$, where a subdomain D_{-1} is made of particulars (extensional entities) and contains all real numbers, and the rest $D_I = D_0 \cup D_1 \dots \cup D_n \dots$ is made of universals (D_0 for propositions (the 0-ary concepts), and $D_n, n \geq 1$, for n-ary concepts). So that, similarly to Boolean algebra for classic logic, we have the Intensional algebra in Definition 7 $Alg_{int} = (\mathcal{D}, f, t, Id, Truth, \{conj_S\}_{S \in \mathcal{P}(\mathbb{N}^2)}, neg, \{exists_n\}_{n \in \mathbb{N}})$, with binary operations $conj_S : D_I \times D_I \rightarrow D_I$, unary operation $neg : D_I \rightarrow D_I$, unary operations $exists_n : D_I \rightarrow D_I$. The sets $f = \emptyset$, $t = \{\langle\rangle\}$ are the empty set and the set with empty tuple $\langle\rangle \in D_{-1}$ (i.e., the unique tuple of 0-ary

relation) used for logic falsity and truth as in Codd's relational-database algebra [21], while $Truth \in D_0$ is the concept (intension) of the tautology. We define that $\mathcal{D}^0 = \{\langle\rangle\}$, so that $\{f, t\} = \mathcal{P}(\mathcal{D}^0)$.

Any extensionalization function $h \in \mathcal{E}$ assigns to the carrier set \mathcal{D}_I of this algebra (i.e., universals or concepts) an appropriate extension as follows: for each proposition $u \in D_0$, $h(u) \in \{f, t\} = \mathcal{P}(\mathcal{D}^0) = \mathcal{P}(\{\langle\rangle\}) \subset \mathcal{P}(D_{-1})$ is its extension (true or false value), where \mathcal{P} is the powerset operator, and $h(Truth) = t$; for each n-ary concept $u \in D_n$, $h(u)$ is a subset of \mathcal{D}^n ; in the case of particulars $u \in D_{-1}$, we have that $h(u) = u$. We require that operations *conj*, *disj* and *neg* in this intensional algebra behave in the expected way with respect to each extensionalization function (for example, for all $u \in D_0$, $h(neg(u)) = t$ iff $h(u) = f$, etc.), that is

$$h = h_{-1} + \sum_{i \geq 0} h_i : \sum_{i \geq -1} D_i \longrightarrow D_{-1} + \sum_{i \geq 0} \mathcal{P}(\mathcal{D}^i)$$

where $h_{-1} = id : D_{-1} \rightarrow D_{-1}$ is the identity, $h_0 : D_0 \rightarrow \{f, t\}$ assigns the truth values in $\{f, t\}$, to all propositions, and $h_i : D_i \rightarrow \mathcal{P}(\mathcal{D}^i)$, $i \geq 1$, assigns an extension to all concepts. Thus, the intensions can be seen as *names* of abstract or concrete entities, while the extensions correspond to various rules that these entities play in different worlds: as shown in Appendix by Bealer-Montague isomorphism, each extensionalization function h can be considered equivalently as a "possible world" in Montague's semantics for intensional logic. The Tarski-style definitions of truth and validity for this intensional FOL language \mathbb{L} may be given in the customary way. An *intensional interpretation* is a mapping between the set \mathbb{L} of formulae of the logic language and intensional entities in \mathcal{D} , $I : \mathbb{L} \rightarrow \mathcal{D}$, is a kind of "conceptualization", such that an open-sentence (virtual predicate) $\phi(x_1, \dots, x_k) \in \mathbb{L}$ with a tuple of all free variables (x_1, \dots, x_k) is mapped into a k-ary *concept*, that is, an intensional entity $u = I(\phi(x_1, \dots, x_k)) \in D_k$, and (closed) sentence ψ into a proposition (i.e., *logic* concept) $u = I(\psi) \in D_0$ with $I(\top) = Truth \in D_0$ for the FOL tautology \top . A language constant c is mapped into a particular $\bar{c} = I(c) \in D_{-1}$ if it is a proper name, otherwise in a correspondent concept in \mathcal{D} . The interpretations of singular abstracted terms $\langle\phi\rangle_{x_1 \dots x_m}$ will denote an appropriate proposition (for $m = 0$), property (for $m = 1$), or relation (for $m \geq 2$). Thus, the mapping of intensional abstracts (terms) into \mathcal{D} will be an extension of I to all abstract, such that the interpretation of $\langle\phi\rangle$ is equal to the meaning of a proposition $\phi \in \mathbb{L}$, that is, $I(\langle\phi\rangle) = I(\phi) \in D_0$. In the case when ϕ is an atom $p_i^m(x_1, \dots, x_m) \in \mathbb{L}$ then $I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}) = I(p_i^m(x_1, \dots, x_m)) \in D_m$.

The basic intensional logic language $L_{PR} \subseteq L$ for probabilistic theory is composed by propositions in $L(\Phi)$, with propositional symbols (0-ary predicate symbols) $p_i^0 = p_i \in \Phi$ (with $I(p_i) \in D_0$), with the binary predicate p_2^3 for the weight or probabilistic function w_N , the binary built-in (with constant fixed extension in any "world" $h \in \mathcal{E}$) predicate p_2^2 for \leq (the binary predicate = for identity is defined by $a = b$ iff $a \leq b$ and $b \leq a$), and two built-in ternary predicates p_3^1, p_3^3 , denoted by \oplus, \odot , for addition and multiplication operations $+, \cdot$ respectively as required for a logic for reasoning about probabilities [7]. The 0-ary functional symbols a, b, c, \dots in this logic language will be used as numeric constants for denotation of probabilities in $[0, 1]$, i.e., with $I(a) = \bar{a} \in [0, 1] \subset D_{-1}$.

Consequently, the "worlds" (i.e., the extensionalization functions) will be reduced to the mappings

$$h = h_{-1} + h_0 + h_2 + h_3.$$

We recall that in intensional FOL each n-ary functional symbol is represented by the (n+1)-ary predicate letter, where the last attribute is introduced for the resulting values of such a function. For example, the first attribute of the predicate letter w_N will contain the intensional abstract of a propositional formula in $L(\Phi)$, while the second place will contain the probabilistic value in the interval of reals $[0, 1] \subset D_{-1}$, so that the ground atom $w_N(\langle \phi \rangle, a)$ in L_{PR} will have the interpretation $I(w_N(\langle \phi \rangle, a)) \in D_0$. The atom $w_N(x, y)$, with variables x and y , will satisfy the functional requirements, that is $I(w_N(x, y)) \in D_2$ with a binary relation $R = h(I(w_N(x, y))) \in \mathcal{P}(D_0 \times [0, 1]) \subseteq \mathcal{P}(D^2)$, such that for any $(u, v) \in R$ there is no $v_1 \neq v$ such that $(u, v_1) \in R$. Obviously, for this intensional logic we have that $h(I(w_N(\langle \phi \rangle, a))) = t$ iff $(I(\langle \phi \rangle), I(a)) = (I(\phi), \bar{a}) \in R$.

Analogously, for the ground atom $\oplus(a, b, c)$, with $\bar{a} = I(a), \bar{b} = I(b), \bar{c} = I(c) \in D_{-1}$ real numbers, we have that $I(\oplus(a, b, c)) \in D_0$ such that for any "world" $h \in \mathcal{E}$ we have that $h(I(\oplus(a, b, c))) = t$ iff $\bar{a} + \bar{b} = \bar{c}$ (remember that for elements $\bar{a} = I(a) \in D_{-1}$ we have that $h(I(a)) = \bar{a}$). For addition of more than two elements in this intensional logic we will use intensional abstract, for example for the sum of three elements we can use a ground formula $\oplus(a, b, d) \wedge \oplus(d, c, e)$, such that it holds that $h(I(\oplus(a, b, d) \wedge \oplus(d, c, e))) = \text{conj}(I(\oplus(a, b, d)), I(\oplus(d, c, e))) = t$ iff $\bar{a} + \bar{b} = \bar{d}$ and $\bar{d} + \bar{c} = \bar{e}$, that is, iff $\bar{a} + \bar{b} + \bar{c} = \bar{e}$. The fixed extensions of the two built-in ternary predicates $\oplus(x, y, z)$ and $\odot(x, y, z)$ are equal to:

$$R_{\oplus} = h(I(\oplus(x, y, z))) = \{(u_1, u_2, u_1 + u_2) \mid u_1, u_2 \in D_{-1} \text{ are real numbers}\},$$

$$R_{\odot} = h(I(\odot(x, y, z))) = \{(u_1, u_2, u_1 \cdot u_2) \mid u_1, u_2 \in D_{-1} \text{ are real numbers}\}.$$

The built-in binary predicate \leq satisfies the following requirements for its intensional interpretation: $I(\leq(x, y)) \in D_2$ such that for every $h \in \mathcal{E}$ it holds that its fixed extension is a binary relation $R_{\leq} = h(I(\leq(x, y))) = \{(u, v) \mid u, v \in D_{-1} \text{ are real numbers and } u \leq v\}$, with the property that $h(I(\leq(a_1, a_2))) = t$ iff $(I(a_1), I(a_2)) = (\bar{a}_1, \bar{a}_2) \in R_{\leq}$.

Definition 4 *Intensional FOL L_{PR} is a probabilistic logic with a probability structure $M = (\mathcal{Z}^{\Phi}, \mathcal{P}(\mathcal{Z}^{\Phi}), \mu, \pi)$ if its intentional interpretations satisfy the following property for any propositional formula $\phi \in L(\Phi) \subseteq L_{PR}$: $h(I(w_N(\langle \phi \rangle, a))) = t$ iff $I(a) = \sum_{s \in \mathcal{Z}^{\Phi} \ \& \ \pi(s)(\phi)=1} \mu(\{s\})$.*

Let us show that the binary predicate w_N is a functional built-in predicate, whose extension is equal in every possible "world" $h \in \mathcal{E}$, and that the probability structure can use \mathcal{E} as the set of possible worlds in the place of Nilsson's set \mathcal{Z}^{Φ} . That is, we can replace Nilsson's structure with the intensional probability structure $M_I = (\mathcal{E}, \mathcal{P}(\mathcal{E}), \mu, \pi)$.

Proposition 2 *Intensional FOL L_{PR} is a probabilistic logic with a probability structure $M = (\mathcal{Z}^{\Phi}, \mathcal{P}(\mathcal{Z}^{\Phi}), \mu, \pi)$ if w_N is a built-in functional symbol such that its fixed extension is equal to:*

$$R_{w_N} = h(I(w_N(x, y))) = \{(I(\phi), I(a)) \mid \phi \in L(\Phi) \text{ and } I(a) = \sum_{h_1 \in \mathcal{E} \ \& \ h_1(I(w_N(\langle \phi \rangle, a)))=t} \mu(\{is^{-1}(h_1)\})\}$$

where the mapping $is : \mathcal{Z}^{\Phi} \rightarrow \mathcal{E}$ is a bijection, and is^{-1} its inverse.

Proof: Let us show that there is a bijection $is : \mathcal{Z}^{\Phi} \rightarrow \mathcal{E}$ between the sets \mathcal{Z}^{Φ} and \mathcal{E} . In fact, let $v \in \mathcal{Z}^{\Phi}$ be extended (in the unique standard homomorphic way) to all propositional formulae by $\bar{v} : L(\Phi) \rightarrow \mathbf{2}$. This propositional valuation corresponds to the intensional interpretation (I, h) obtained, for any sentence $\phi \in L(\Phi)$, by $h(I(\phi)) = is_2(\bar{v}(\phi))$, where $is_2 : \mathbf{2} \rightarrow \{f, t\}$ is a bijection of these two lattices such that $is_2(0) = f$, $is_2(1) = t$. We have seen that all predicate symbols with arity bigger than 0 of our intensional probabilistic logic L_{PR} are *built-in predicates* (that do not depend on $h \in \mathcal{E}$), so that for a fixed intensional interpretation I , any two extensionalization functions h, h' differ only on propositions in D_0 , so that we obtain the bijective mapping $is : v \mapsto h$, such that $v = is_2^{-1} \circ h \circ I$, where \circ denotes the composition of functions.

From Tarski's constraint (T) of intensional algebra (in Appendix) we have that for any ground atom $w_N(\langle \phi \rangle, a)$ it holds that:

$$I(w_N(\langle \phi \rangle, a)) = t \text{ iff } (I(\phi), I(a)) \in h(I(w_N(x, y))).$$

Thus, from Definition 4, we obtain that $(I(\phi), I(a)) \in h(I(w_N(x, y)))$ iff $I(a) = \sum_{s \in \mathcal{Z}^{\Phi} \ \& \ \pi(s)(\phi)=1} \mu(\{s\})$.

Consequently,

$R_{w_N} = h(I(w_N(x, y))) = \{(I(\phi), I(a)) \mid \phi \in L(\Phi) \text{ and } I(a) = \sum_{s \in 2^\Phi \text{ \& } \pi(s)(\phi)=1} \mu(\{s\})\}$, where $I(w_N(x, y)) \in D_2$, $I(\phi) \in D_0$ and $I(a) \in [0, 1] \subseteq D_{-1}$. But from a bijection is , instead of $s \in 2^\Phi$ we can take $h_1 = is(s) \in \mathcal{E}$, and the fact $\pi(s)(\phi) = 1$ means that " ϕ is true in the state s " can be equivalently replaced by " ϕ is true in the "world" $h_1 = is(s)$ ", that is by condition $h_1(I(w_N(\langle \phi \rangle, a))) = t$, so that definition for the extension of the binary relation R_{w_N} for Nilsson's probabilities of propositional formulae, given in this proposition, is correct. This extension is constant in any "possible world" in \mathcal{E} , thus, the binary functional-predicate w_N is a built-in predicate in this intensional FOL L_{PR} .

□

Consequently, the sentence "the probability that ϕ is equal to a ", expressed by the ground atom $w_N(\langle \phi \rangle, a)$, is true iff $h(I(w_N(\langle \phi \rangle, a))) = t$ iff $(I(\phi), \bar{a}) \in h(I(w_N(x, y))) = R_{w_N}$.

Thus, for the most simple linear inequality, "the probability that ϕ is less then or equal to c ", expressed by the formula $\exists x(w_N(\langle \phi \rangle, x) \wedge \leq (x, c))$, is true iff $h(I(\exists x(w_N(\langle \phi \rangle, x) \wedge \leq (x, c)))) = t$ iff $(u, I(c)) \in R_{\leq}$ where $u \in [0, 1] \subset D_{-1}$ is determined by $(v, u) \in R_{w_N}$ where $v = I(\phi) \in D_0$.

Analogously to the results obtained for a logic for reasoning about probabilities in [7], we obtain the following property:

Theorem 1 *The intensional FOL L_{PR} with built-in binary predicate w_N defined in Proposition 2, built-in binary predicate \leq and ternary built-in predicates \oplus, \odot , is sound and complete with respect to the measurable probability structures.*

Proof: We will follow the demonstration analogous to the demonstration of Theorem 2.2 in [7] for the sound and complete axiomatization of the axiomatic system AX_{MEAS} for logic reasoning about probabilities, divided into three parts, which deal respectively with propositional reasoning, reasoning about linear inequalities, and reasoning about probabilities:

1. Propositional reasoning: set of all instances of propositional tautologies, with unique inference rule Modus Ponens.

2. Reasoning about linear inequalities: set of all instances of valid formulae about linear inequalities of the form $a_1 \cdot x_1 + \dots + a_k \cdot x_k \leq c$, where a_1, \dots, a_k and c are integers with $k \geq 1$, while x_1, \dots, x_k are probabilistic variables.

3. Reasoning about probability function:

3.1 $w(\phi) \geq 0$ (nonnegativity)

3.2 $w(true) = 1$ (the probability of the event true is 1)

3.3 $w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w(\phi)$ (additivity)

3.4 $w(\phi) = w(\psi)$ if $\phi \equiv \psi$ (distributivity).

It is easy to verify that for any propositional axiom ϕ , we have that for all "worlds" $h \in \mathcal{E}$ it holds that $h(I(\phi)) = t$, so that it is true in the S5 Kripke model of the intensional FOL given in Definition 9 (Appendix), because all algebraic operations in Alg_{int} in Definition 7(Appendix) are defined in order to satisfy standard propositional logic. Moreover, the Modus Ponens rule is satisfied in every "world" $h \in \mathcal{E}$. Thus, the point 1 above is satisfied by intensional logic L_{PR} .

The definition of built-in predicates \odot, \oplus and \leq satisfy all linear inequalities, thus the point 2 above.

The definition of binary predicate $w_N(x, y)$ is given in order to satisfy Nilsson's probability structure, thus all properties of probability function in the point 3 above are satisfied by $w_N(x, y)$ built-in predicate in every "world" $h \in \mathcal{E}$. Consequently, the soundness and completeness of the intensional logic L_{PR} with respect to measurable probability structures, based on the Theorem Theorem 2.2 in [7] is satisfied.

□

For example, the satisfaction of the linear inequality $a_1 \cdot x_1 + a_2 \cdot x_2 \leq c$, where x_1 and x_2 are the probabilities of the propositional formulae ϕ_1 and ϕ_2 relatively, (here the list of quantifications $(\exists x_1) \dots (\exists x_k)$ is abbreviated by $(\exists x_1, \dots, x_k)$), expressed by the following intensional formula,

$(\exists x_1, x_2, y_1, y_2, y_3)(w_N(\langle \phi_1 \rangle, x_1) \wedge w_N(\langle \phi_2 \rangle, x_2) \wedge \odot(a_1, x_1, y_1) \wedge \odot(a_2, x_2, y_2) \wedge \oplus(y_1, y_2, y_3) \wedge \leq (y_3, c))$, is true iff $(I(\phi_1), u_1), (I(\phi_2), u_2) \in R_{w_N}$, $(I(a_1), u_1, v_1), (I(a_2), u_2, v_2) \in R_{\odot}$, $(v_1, v_2, v_3) \in R_{\oplus}$ and $(v_3, I(c)) \in R_{\leq}$,

where $u_1, u_2, v_1, \dots, v_3 \in D_{-1}$ are real numbers.

Analogously, the satisfaction of any linear inequality $a_1 \cdot x_1 + \dots + a_k \cdot x_k \leq c$, where x_i are the probabilities of the propositional formulae ϕ_i for $i = 1, \dots, k$, $k \geq 2$, can be expressed by the logic formula

$(\exists x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k)(w_N(\langle \phi_1 \rangle, x_1) \wedge \dots \wedge w_N(\langle \phi_k \rangle, x_k) \wedge \odot(a_1, x_1, y_1) \wedge \dots \wedge \odot(a_k, x_k, y_k) \wedge \oplus(0, y_1, z_1) \wedge \oplus(z_1, y_2, z_2) \wedge \dots \wedge \oplus(z_{k-1}, y_k, z_k) \wedge \leq (z_k, c))$.

Based on the Theorem 2.9 in [7] we can conclude that the problem of deciding whether such an intensional formula in L_{PR} is satisfiable in a measurable probability structure of Nilsson is NP-complete.

4 Application to Probabilistic Logic Programs

The semantics of the interval-based Probabilistic Logic Programs, based on possible worlds with the fixpoint semantics for such programs [1], has been

considered valid for more than 13 years. But some years ago, when I worked with Prof. Subrahmanian director of the UMIACS institute, I have had the opportunity to consider the general problems of (temporal) probabilistic databases [22], to analyze their semantics of interval-based Probabilistic Logic Programs, and to realize that unfortunately it was not correctly defined.

Because of that I formally developed, in [23], the reduction of (temporal) probabilistic databases into Constraint Logic Programs. Consequently, it was possible to apply the interval PSAT in order to find the models of such interval-based probabilistic programs, as presented and compared with other approaches in [4]. Moreover, in this complete revision it was demonstrated that the Temporal-Probabilistic Logic Programs can be reduced into the particular case of the ordinary Probabilistic Logic Programs, so that our application of intensional semantics we can apply only to this last general case of Logic Programs.

In what follows I will briefly introduce the syntax of Probabilistic Logic Programs. More about it can be found in the original work in [1] and in its last revision in [4]. Let $ground(P)$ denote the set of all ground instances of rules of a Probabilistic Logic Program P with a given domain for object variables, and let H denote the Herbrand base of this program P . Then, each ground instance of rules in $ground(P)$ has the following syntax:

$$(*) \quad A : \mu_0 \leftarrow \phi_1 : \mu_1 \wedge \dots \wedge \phi_m : \mu_m,$$

where $A \in H$ is a ground atom in a Herbrand base H , $\phi_i, i \geq 1$ are logic formulae composed by ground atoms and standard logic connectives \wedge and \neg , while $\mu_i = (b_i, c_i), i \geq 0$, where $b_i, c_i \in [0, 1]$, are the lower and upper probability boundaries.

The expression $\phi_i : \mu_i$ is a probabilistic-annotated (p-annotated) basic formula, which is true if the probability x_i of the ground formula ϕ_i is between b_i and c_i ; false otherwise. Thus, this basic p-annotated formula is the particular case of the 2-valued probabilistic formula:

$$(**) \quad (1 \cdot x_i \geq a_i) \wedge (1 \cdot x_i \leq b_i)$$

composed by two linear inequalities.

Consequently, the standard logic embedding of annotated interval-based logic programs can be easily obtained by the intensional logic L_{PR} described in Section 3 where Φ is equal to the Herbrand base H of the annotated interval-based probabilistic logic program P .

Thus, based on the translation (**), the logic formula in intensional logic L_{PR} correspondent to basic annotated formula $\phi_i : \mu_i$ of the annotated logic program $ground(P)$, is equal to the following first-order closed

formulae with a variable x_i :

$$\exists x_i (w_N(\leq \phi_i, x_i) \wedge \leq (b_i, x_i) \wedge \leq (x_i, c_i)).$$

Based on this translation, the rule $(*)$ of the annotated logic Program $ground(P)$ can be replaced by the following rule of an intensional probabilistic logic program:

$$\exists x_0 (w_N(\leq A, x_0) \wedge \leq (b_0, x_0) \wedge \leq (x_0, c_0)) \leftarrow$$

$$\leftarrow \exists x_1 (w_N(\leq \phi_1, x_1) \wedge \leq (b_1, x_1) \wedge \leq (x_1, c_1)) \wedge \dots$$

$$\wedge \exists x_m (w_N(\leq \phi_m, x_m) \wedge \leq (b_m, x_m) \wedge \leq (x_m, c_m)),$$

with the variables x_0, x_1, \dots, x_m .

In this way we obtain a grounded intensional probabilistic logic program P_{PR} , which has both the syntax and semantics different from the original annotated probabilistic logic program $ground(P)$.

As an alternative to this full intensional embedding of the annotated logic programs into the first-order intensional logic, we can use a partial embedding by preserving the old ad hoc annotated syntax of the probabilistic program $ground(P)$, by extending the standard predicate-based syntax of the intensional FOL logic with annotated formulae, and by defining only the new intensional interpretation I for these annotated formulae, as follows: $I(\phi_i : \mu_i) =$

$$= I((\exists x)(w_N(\leq \phi_i, x) \wedge (\leq (b_i, x) \wedge \leq (x, c_i))))$$

$$= \text{exist}(\text{conj}_{\{(1,1)\}}(I(w_N(\leq \phi_i, x)), \text{conj}_{\{(1,1)\}}(I(\leq (b_i, x)), I(\leq (x, c_i)))))$$

where $I(w_N(\leq \phi_i, x)), I(\leq (b_i, x)), I(\leq (x, c_i)) \in D_1$.

So that $h(I(\phi_i : \mu_i)) = t$ iff $h(\text{exist}(u_1)) = t$, where

$$u_1 = \text{conj}_{\{(1,1)\}}(I(w_N(\leq \phi_i, x)), \text{conj}_{\{(1,1)\}}(I(\leq (b_i, x)), I(\leq (x, c_i)))) \in D_1,$$

iff $\exists u(u \in h(u_1))$ iff $(I(b_i), u), (u, I(c_i)) \in R_{\leq}$,

where $u \in [0, 1] \subset D_{-1}$ is a particular assignment for a variable x determined by $(v, u) \in R_{w_N}$ where $v = I(\phi_i)$.

Notice that, from Appendix and the fact that $u_1 \in D_1$,

$$h(\text{exist}(u_1)) = f_{<>}(h(I(w_N(\leq \phi_i, x)) \bowtie_{\{(1,1)\}} h(I(\leq (b_i, x))) \bowtie_{\{(1,1)\}} h(I(\leq (x, c_i)))))$$

$$= f_{<>}(h(I(w_N(\leq \phi_i, x)) \cap h(I(\leq (b_i, x))) \cap h(I(\leq (x, c_i))))), \text{ where } f_{<>}(R) = f \text{ if } R = \emptyset; t \text{ otherwise.}$$

The advantage of this second, partial embedding is that we can preserve the old syntax for (temporal) probabilistic logic programs [1, 4], but providing to them the standard intensional FOL semantics instead of current ad hoc semantics for such a kind of logic programs.

5 Conclusion

There are two consequences of this full and natural embedding of the probability theory in a logic:

1. The full embedding of Nilsson' probabilistic structure into propositional logic results in a *truth-functional* many-valued logic, differently from

Nilsson's intuition and current considerations about propositional probabilistic logic.

2. The logic for reasoning *about* probabilities can be embedded into an intensional FOL that remains to be *2-valued logic*, both for propositional formulae in $L(\Phi)$ and predicate formulae for probability constraints, based on the binary built-in predicate \leq and binary predicate w_N used for the probability function, where the basic propositional letters in Φ are formally considered as nullary predicate symbols.

The intensional FOL for reasoning about probabilities is obtained by the particular fusion of the intensional algebra (analogously to Bealer's approach) and Montague's possible-worlds modal logic for the semantics of the natural language. In this paper we enriched such a logic framework by a number of built-in binary and ternary predicates, which can be used to define the basic set of probability inequalities and to render the probability weight function w_N an explicit object in this logic language. We conclude that this intensional FOL logic with intensional abstraction is a good candidate language for specification of Probabilistic Logic Programs, and we apply two different approaches: first one is obtained by the translation of the annotated syntax of current logic programs into this intensional FOL; the second one, instead, modify only the semantics of these logic programs by preserving their current ad-hoc annotated syntax.

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6 Appendix: Intensional FOL language with intensional abstraction

Intensional entities are such concepts as propositions and properties. What make them ‘intensional’ is that they violate the principle of extensionality; the principle that extensional equivalence implies identity. All (or most) of these intensional entities have been classified at one time or another as kinds of Universals [24]. We consider a non empty domain $\mathcal{D} = D_{-1} \cup D_I$, where a subdomain D_{-1} is made of particulars (extensional entities), and the rest $D_I = D_0 \cup D_1 \dots \cup D_n \dots$ is made of universals (D_0 for propositions (the 0-ary concepts), and $D_n, n \geq 1$, for n-ary concepts). The fundamental entities are *intensional abstracts* or so called, ‘that’-clauses. We assume that they are singular terms; Intensional expressions like ‘believe’, ‘mean’, ‘assert’, ‘know’, are standard two-place predicates that

take 'that'-clauses as arguments. Expressions like 'is necessary', 'is true', and 'is possible' are one-place predicates that take 'that'-clauses as arguments. For example, in the intensional sentence "it is necessary that ϕ ", where ϕ is a proposition, the 'that ϕ ' is denoted by the $\langle\phi\rangle$, where $\langle\rangle$ is the intensional abstraction operator which transforms a logic formula into a *term*. Or, for example, "x believes that ϕ " is given by formula $p_i^2(x, \langle\phi\rangle)$ (p_i^2 is binary 'believe' predicate). Here we will present an intensional FOL with slightly different intensional abstraction than that originally presented in [25]:

Definition 5 *The syntax of the First-order Logic language with intensional abstraction $\langle\rangle$, denoted by L , is as follows:*

Logic operators (\wedge, \neg, \exists); Predicate letters in P (functional letters are considered as particular case of predicate letters); Variables x, y, z, \dots in \mathcal{V} ; Abstraction $\langle\rangle$, and punctuation symbols (comma, parenthesis). With the following simultaneous inductive definition of term and formula :

1. *All variables and constants (0-ary functional letters in P) are terms.*
2. *If t_1, \dots, t_k are terms, then $p_i^k(t_1, \dots, t_k)$ is a formula ($p_i^k \in P$ is a k -ary predicate letter).*
3. *If ϕ and ψ are formulae, then $(\phi \wedge \psi)$, $\neg\phi$, and $(\exists x)\phi$ are formulae.*
4. *If $\phi(\mathbf{x})$ is a formula (virtual predicate) with a list of free variables in $\mathbf{x} = (x_1, \dots, x_n)$ (with ordering from-left-to-right of their appearance in ϕ), and α is its sublist of distinct variables, then $\langle\phi\rangle_\alpha^\beta$ is a term, where β is the remaining list of free variables preserving ordering in \mathbf{x} as well. The externally quantifiable variables are the free variables not in α . When $n = 0$, $\langle\phi\rangle$ is a term which denotes a proposition, for $n \geq 1$ it denotes a n -ary concept.*

An occurrence of a variable x_i in a formula (or a term) is bound (free) iff it lies (does not lie) within a formula of the form $(\exists x_i)\phi$ (or a term of the form $\langle\phi\rangle_\alpha^\beta$ with $x_i \in \alpha$). A variable is free (bound) in a formula (or term) iff it has (does not have) a free occurrence in that formula (or term).

A sentence is a formula having no free variables. The binary predicate letter p_1^2 for identity is singled out as a distinguished logical predicate and formulae of the form $p_1^2(t_1, t_2)$ are to be rewritten in the form $t_1 \doteq t_2$. We denote by $R_=\$ the binary relation obtained by standard Tarski's interpretation of this predicate p_1^2 . The logic operators $\forall, \vee, \Rightarrow$ are defined in terms of (\wedge, \neg, \exists) in the usual way.

The *intensional interpretation* of this intensional FOL is a mapping between the set L of formulae of the logic

language and intensional entities in \mathcal{D} , $I : L \rightarrow \mathcal{D}$, is a kind of "conceptualization", such that an open-sentence (virtual predicate) $\phi(x_1, \dots, x_k)$ with a tuple of all free variables (x_1, \dots, x_k) is mapped into a k -ary *concept*, that is, an intensional entity $u = I(\phi(x_1, \dots, x_k)) \in D_k$, and (closed) sentence ψ into a proposition (i.e., *logic concept*) $v = I(\psi) \in D_0$ with $I(\top) = \text{Truth} \in D_0$ for the FOL tautology \top . A language constant c is mapped into a particular $a = I(c) \in D_{-1}$ if it is a proper name, otherwise in a correspondent concept in \mathcal{D} .

An assignment $g : \mathcal{V} \rightarrow \mathcal{D}$ for variables in \mathcal{V} is applied only to free variables in terms and formulae. Such an assignment $g \in \mathcal{D}^\mathcal{V}$ can be recursively uniquely extended into the assignment $g^* : \mathcal{T} \rightarrow \mathcal{D}$, where \mathcal{T} denotes the set of all terms (here I is an intensional interpretation of this FOL, as explained in what follows), by :

1. $g^*(t) = g(x) \in \mathcal{D}$ if the term t is a variable $x \in \mathcal{V}$.
2. $g^*(t) = I(c) \in \mathcal{D}$ if the term t is a constant $c \in P$.
3. if t is an abstracted term $\langle\phi\rangle_\alpha^\beta$, then $g^*(\langle\phi\rangle_\alpha^\beta) = I(\phi[\beta/g(\beta)]) \in D_k, k = |\alpha|$ (i.e., the number of variables in α), where $g(\beta) = g(y_1, \dots, y_m) = (g(y_1), \dots, g(y_m))$ and $[\beta/g(\beta)]$ is a uniform replacement of each i -th variable in the list β with the i -th constant in the list $g(\beta)$. Notice that α is the list of all free variables in the formula $\phi[\beta/g(\beta)]$.

We denote by t/g (or ϕ/g) the ground term (or formula) without free variables, obtained by assignment g from a term t (or a formula ϕ), and by $\phi[x/t]$ the formula obtained by uniformly replacing x by a term t in ϕ .

The distinction between intensions and extensions is important especially because we are now able to have and *equational theory* over intensional entities (as $\langle\phi\rangle$), that is predicate and function "names", that is separate from the extensional equality of relations and functions. An extensionalization function h assigns to the intensional elements of \mathcal{D} an appropriate extension as follows: for each proposition $u \in D_0$, $h(u) \in \{f, t\} \subseteq \mathcal{P}(D_{-1})$ is its extension (true or false value); for each n -ary concept $u \in D_n$, $h(u)$ is a subset of \mathcal{D}^n (n -th Cartesian product of \mathcal{D}); in the case of particulars $u \in D_{-1}$, $h(u) = u$.

The sets f, t are empty set $\{\}$ and set $\{\langle\rangle\}$ (with the empty tuple $\langle\rangle \in D_{-1}$ i.e. the unique tuple of 0-ary relation) which may be thought of as falsity and truth, as those used in the Codd's relational-database algebra [21] respectively, while $\text{Truth} \in D_0$ is the concept (intension) of the tautology.

We define that $\mathcal{D}^0 = \{\langle\rangle\}$, so that $\{f, t\} = \mathcal{P}(\mathcal{D}^0)$. Thus we have:

$$h = h_{-1} + \sum_{i \geq 0} h_i : \sum_{i \geq -1} D_i \longrightarrow D_{-1} + \sum_{i \geq 0} \mathcal{P}(D^i),$$

where $h_{-1} = id : D_{-1} \rightarrow D_{-1}$ is identity, $h_0 : D_0 \rightarrow \{f, t\}$ assigns truth values in $\{f, t\}$, to all propositions, and $h_i : D_i \rightarrow \mathcal{P}(D^i)$, $i \geq 1$, assigns extension to all concepts, where \mathcal{P} is the powerset operator. Thus, intensions can be seen as *names* of abstract or concrete entities, while extensions correspond to various rules that these entities play in different worlds.

Remark: (Tarski's constraint) This semantics has to preserve Tarski's semantics of the FOL, that is, for any formula $\phi \in \mathcal{L}$ with the tuple of free variables (x_1, \dots, x_k) , any assignment $g \in \mathcal{D}^V$, and every $h \in \mathcal{E}$ it has to be satisfied that:

(T) $h(I(\phi/g)) = t$ iff $(g(x_1), \dots, g(x_k)) \in h(I(\phi))$.

□

Thus, intensional FOL has the simple Tarski first-order semantics, with a decidable unification problem, but we need also the actual world mapping which maps any intensional entity to its *actual world extension*. In what follows we will identify a *possible world* by a particular mapping which assigns to intensional entities their extensions in such possible world. That is direct bridge between intensional FOL and possible worlds representation [26, 27, 28, 29, 30], where intension of a proposition is a *function* from possible worlds $\tilde{\mathcal{W}}$ to truth-values, and properties and functions from $\tilde{\mathcal{W}}$ to sets of possible (usually not-actual) objects.

Here \mathcal{E} denotes the set of possible *extensionalization functions* that satisfy the constraint (T); they can be considered as *possible worlds* (as in Montague's intensional semantics for natural language [28, 30]), as demonstrated in [31, 12], given by the bijection $is : \mathcal{W} \simeq \mathcal{E}$.

Now we are able to define formally this intensional semantics:

Definition 6 TWO-STEP INTENSIONAL SEMANTICS: Let $\mathfrak{R} = \bigcup_{k \in \mathbb{N}} \mathcal{P}(\mathcal{D}^k) = \sum_{k \in \mathbb{N}} \mathcal{P}(\mathcal{D}^k)$ be the set of all k -ary relations, where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Notice that $\{f, t\} = \mathcal{P}(\mathcal{D}^0) \in \mathfrak{R}$, that is, the truth values are extensions in \mathfrak{R} .

The intensional semantics of the logic language with the set of formulae \mathcal{L} can be represented by the mapping

$$\mathcal{L} \xrightarrow{I} \mathcal{D} \Rightarrow_{w \in \mathcal{W}} \mathfrak{R},$$

where \xrightarrow{I} is a fixed intensional interpretation $I : \mathcal{L} \rightarrow \mathcal{D}$ and $\Rightarrow_{w \in \mathcal{W}}$ is the set of all extensionalization functions $h = is(w) : \mathcal{D} \rightarrow \mathfrak{R}$ in \mathcal{E} , where $is : \mathcal{W} \rightarrow \mathcal{E}$ is the mapping from the set of possible worlds to the set of extensionalization functions.

We define the mapping $I_n : L_{op} \rightarrow \mathfrak{R}^{\mathcal{W}}$, where L_{op} is the subset of formulae with free variables (virtual predicates), such that for any virtual predicate $\phi(x_1, \dots, x_k) \in L_{op}$ the mapping $I_n(\phi(x_1, \dots, x_k)) :$

$\mathcal{W} \rightarrow \mathfrak{R}$ is the Montague's meaning (i.e., intension) of this virtual predicate [26, 27, 28, 29, 30], that is, the mapping which returns with the extension of this (virtual) predicate in every possible world in \mathcal{W} .

We adopted this two-step intensional semantics, instead of well known Montague's semantics (which lies in the construction of a compositional and recursive semantics that covers both intension and extension) because of a number of its weakness.

Example 1: Let us consider the following two past participles: 'bought' and 'sold' (with unary predicates $p_1^1(x)$, 'x has been bought', and $p_2^1(x)$, 'x has been sold'). These two different concepts in the Montague's semantics would have not only the same extension but also their intension, from the fact that their extensions are identical in every possible world. Within the two-steps formalism we can avoid this problem by assigning two different concepts (meanings) $u = I(p_1^1(x))$ and $v = I(p_2^1(x))$ in $\in D_1$. Notice that the same problem we have in the Montague's semantics for two sentences with different meanings, which bear the same truth value across all possible worlds: in the Montague's semantics they will be forced to the *same* meaning.

□

Another relevant question w.r.t. this two-step interpretations of an intensional semantics is how in it is managed the extensional identity relation \doteq (binary predicate of the identity) of the FOL. Here this extensional identity relation is mapped into the binary concept $Id = I(\doteq(x, y)) \in D_2$, such that $(\forall w \in \mathcal{W})(is(w)(Id) = R_{\doteq})$, where $\doteq(x, y)$ (i.e., $p_1^2(x, y)$) denotes an atom of the FOL of the binary predicate for identity in FOL, usually written by FOL formula $x \doteq y$ (here we prefer to distinguish this *formal symbol* $\doteq \in P$ of the built-in identity binary predicate letter in the FOL from the standard mathematical symbol '=' used in all mathematical definitions in this paper). In what follows we will use the function $f_{<>} : \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any $R \in \mathfrak{R}$, $f_{<>}(R) = \{<>\}$ if $R \neq \emptyset$; \emptyset otherwise. Let us define the following set of algebraic operators for relations in \mathfrak{R} :

1. binary operator $\bowtie_S : \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any two relations $R_1, R_2 \in \mathfrak{R}$, the $R_1 \bowtie_S R_2$ is equal to the relation obtained by natural join of these two relations if S is a non empty set of pairs of joined columns of respective relations (where the first argument is the column index of the relation R_1 while the second argument is the column index of the joined column of the relation R_2); otherwise it is equal to the cartesian product $R_1 \times R_2$. For example, the logic formula $\phi(x_i, x_j, x_k, x_l, x_m) \wedge \psi(x_l, y_i, x_j, y_j)$ will be

traduced by the algebraic expression $R_1 \bowtie_S R_2$ where $R_1 \in \mathcal{P}(\mathcal{D}^5)$, $R_2 \in \mathcal{P}(\mathcal{D}^4)$ are the extensions for a given Tarski's interpretation of the virtual predicate ϕ, ψ relatively, so that $S = \{(4, 1), (2, 3)\}$ and the resulting relation will have the following ordering of attributes: $(x_i, x_j, x_k, x_l, x_m, y_i, y_j)$.

2. unary operator $\sim: \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any k-ary (with $k \geq 0$) relation $R \in \mathcal{P}(\mathcal{D}^k) \subset \mathfrak{R}$ we have that $\sim(R) = \mathcal{D}^k \setminus R \in \mathcal{D}^k$, where \setminus is the substraction of relations. For example, the logic formula $\neg\phi(x_i, x_j, x_k, x_l, x_m)$ will be traduced by the algebraic expression $\mathcal{D}^5 \setminus R$ where R is the extensions for a given Tarski's interpretation of the virtual predicate ϕ .
3. unary operator $\pi_{-m}: \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any k-ary (with $k \geq 0$) relation $R \in \mathcal{P}(\mathcal{D}^k) \subset \mathfrak{R}$ we have that $\pi_{-m}(R)$ is equal to the relation obtained by elimination of the m-th column of the relation R if $1 \leq m \leq k$ and $k \geq 2$; equal to $f_{<>}(R)$ if $m = k = 1$; otherwise it is equal to R . For example, the logic formula $(\exists x_k)\phi(x_i, x_j, x_k, x_l, x_m)$ will be traduced by the algebraic expression $\pi_{-3}(R)$ where R is the extensions for a given Tarski's interpretation of the virtual predicate ϕ and the resulting relation will have the following ordering of attributes: (x_i, x_j, x_l, x_m) .

Notice that the ordering of attributes of resulting relations corresponds to the method used for generating the ordering of variables in the tuples of free variables adopted for virtual predicates.

Analogously to Boolean algebras which are extensional models of propositional logic, we introduce an intensional algebra for this intensional FOL as follows.

Definition 7 *Intensional algebra for the intensional FOL in Definition 5 is a structure $Alg_{int} = (\mathcal{D}, f, t, Id, Truth, \{conj_S\}_{S \in \mathcal{P}(\mathbb{N}^2)}, neg, \{exists_n\}_{n \in \mathbb{N}})$, with binary operations $conj_S: D_I \times D_I \rightarrow D_I$, unary operation $neg: D_I \rightarrow D_I$, unary operations $exists_n: D_I \rightarrow D_I$, such that for any extensionalization function $h \in \mathcal{E}$, and $u \in D_k, v \in D_j, k, j \geq 0$,*

1. $h(Id) = R_{=}$ and $h(Truth) = \{<>\}$.
2. $h(conj_S(u, v)) = h(u) \bowtie_S h(v)$, where \bowtie_S is the natural join operation defined above and $conj_S(u, v) \in D_m$ where $m = k + j - |S|$ if for every pair $(i_1, i_2) \in S$ it holds that $1 \leq i_1 \leq k, 1 \leq i_2 \leq j$ (otherwise $conj_S(u, v) \in D_{k+j}$).
3. $h(neg(u)) = \sim(h(u)) = \mathcal{D}^k \setminus (h(u))$, where \sim is the operation defined above and $neg(u) \in D_k$.
4. $h(exists_n(u)) = \pi_{-n}(h(u))$, where π_{-n} is the

operation defined above and $exists_n(u) \in D_{k-1}$ if $1 \leq n \leq k$ (otherwise $exists_n$ is the identity function).

Notice that for $u \in D_0$, $h(neg(u)) = \sim(h(u)) = \mathcal{D}^0 \setminus (h(u)) = \{<>\} \setminus (h(u)) \in \{f, t\}$.

We define a derived operation $union: (\mathcal{P}(D_i) \setminus \emptyset) \rightarrow D_i, i \geq 0$, such that, for any $B = \{u_1, \dots, u_n\} \in \mathcal{P}(D_i)$ we have that $union(\{u_1, \dots, u_n\}) =_{def} u_1$ if $n = 1$; $neg(conj_S(neg(u_1), conj_S(\dots, neg(u_n))\dots))$, where $S = \{(l, l) \mid 1 \leq l \leq i\}$, otherwise.

Then we obtain that for $n \geq 2$:

$$\begin{aligned} h(union(B)) &= h(neg(conj_S(neg(u_1), conj_S(\dots, neg(u_n))\dots))) \\ &= \mathcal{D}^i \setminus ((\mathcal{D}^i \setminus h(u_1)) \bowtie_S \dots \bowtie_S (\mathcal{D}^i \setminus h(u_n))) \\ &= \mathcal{D}^i \setminus ((\mathcal{D}^i \setminus h(u_1)) \cap \dots \cap (\mathcal{D}^i \setminus h(u_n))) \\ &= \bigcup \{h(u_j) \mid 1 \leq j \leq n\} = \bigcup \{h(u) \mid u \in B\}. \end{aligned}$$

Once one has found a method for specifying the interpretations of singular terms of \mathcal{L} (take in consideration the particularity of abstracted terms), the Tarski-style definitions of truth and validity for \mathcal{L} may be given in the customary way. What is being sought specifically is a method for characterizing the intensional interpretations of singular terms of \mathcal{L} in such a way that a given singular abstracted term $\langle\phi\rangle_{\alpha}^{\beta}$ will denote an appropriate property, relation, or proposition, depending on the value of $m = |\alpha|$. Thus, the mapping of intensional abstracts (terms) into \mathcal{D} we will define differently from that given in the version of Bealer [32], as follows:

Definition 8 *An intensional interpretation I can be extended to abstracted terms as follows: for any abstracted term $\langle\phi\rangle_{\alpha}^{\beta}$ we define that,*

$$I(\langle\phi\rangle_{\alpha}^{\beta}) = union(\{I(\phi[\beta/g(\beta)]) \mid g \in \mathcal{D}^{\bar{\beta}}\}),$$

where $\bar{\beta}$ denotes the set of elements in the list β , and the assignments in $\mathcal{D}^{\bar{\beta}}$ are limited only to the variables in $\bar{\beta}$.

Remark: Here we can make the question if there is a sense to extend the interpretation also to (abstracted) terms, because in Tarski's interpretation of FOL we do not have any interpretation for terms, but only the assignments for terms, as we defined previously by the mapping $g^*: \mathcal{T} \rightarrow \mathcal{D}$. The answer is positive, because the abstraction symbol $\langle\cdot\rangle_{\alpha}^{\beta}$ can be considered as a kind of the unary built-in functional symbol of intensional FOL, so that we can apply the Tarskian interpretation to this functional symbol into the fixed mapping $I(\langle\cdot\rangle_{\alpha}^{\beta}): \mathcal{L} \rightarrow \mathcal{D}$, so that for any $\phi \in \mathcal{L}$ we have that $I(\langle\phi\rangle_{\alpha}^{\beta})$ is equal to the application of this function to the value ϕ , that is, to $I(\langle\cdot\rangle_{\alpha}^{\beta})(\phi)$. In such an approach we would introduce also the typed variable X for the formulae in \mathcal{L} , so that the Tarskian assignment for this functional symbol with variable X , with $g(X) = \phi \in \mathcal{L}$, can be given by:

$g^*(\langle _ \rangle_{\alpha}^{\beta}(X)) = I(\langle _ \rangle_{\alpha}^{\beta})(g(X)) = I(\langle _ \rangle_{\alpha}^{\beta})(\phi)$
 $= \langle _ \rangle \in D_{-1}$, if $\bar{\alpha} \cup \bar{\beta}$ is not equal to the set of free variables in ϕ ;
 $= \text{union}(\{I(\phi[\beta/g'(\beta)]) \mid g' \in \mathcal{D}^{\bar{\beta}}\}) \in D_{|\bar{\alpha}|}$, otherwise.
 \square

Notice than if $\beta = \emptyset$ is the empty list, then $I(\langle \phi \rangle_{\alpha}^{\beta}) = I(\phi)$. Consequently, the denotation of $\langle \phi \rangle$ is equal to the meaning of a proposition ϕ , that is, $I(\langle \phi \rangle) = I(\phi) \in D_0$. In the case when ϕ is an atom $p_i^m(x_1, \dots, x_m)$ then $I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}) = I(p_i^m(x_1, \dots, x_m)) \in D_m$, while $I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}) = \text{union}(\{I(p_i^m(g(x_1), \dots, g(x_m))) \mid g \in \mathcal{D}^{\{x_1, \dots, x_m\}}\}) \in D_0$, with $h(I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m})) = h(I((\exists x_1) \dots (\exists x_m) p_i^m(x_1, \dots, x_m))) \in \{f, t\}$. For example, $h(I(\langle p_1^1(x_1) \wedge \neg p_1^1(x_1) \rangle_{x_1})) = h(I((\exists x_1)(\langle p_1^1(x_1) \wedge \neg p_1^1(x_1) \rangle_{x_1}))) = f$.

The interpretation of a more complex abstract $\langle \phi \rangle_{\alpha}^{\beta}$ is defined in terms of the interpretations of the relevant syntactically simpler expressions, because the interpretation of more complex formulae is defined in terms of the interpretation of the relevant syntactically simpler formulae, based on the intensional algebra above. For example, $I(p_i^1(x) \wedge p_k^1(x)) = \text{conj}_{\{(1,1)\}}(I(p_i^1(x)), I(p_k^1(x)))$, $I(\neg \phi) = \text{neg}(I(\phi))$, $I(\exists x_i) \phi(x_i, x_j, x_i, x_k) = \text{exists}_3(I(\phi))$.

Consequently, based on the intensional algebra in Definition 7 and on intensional interpretations of abstracted term in Definition 8, it holds that the interpretation of any formula in \mathbf{L} (and any abstracted term) will be reduced to an algebraic expression over interpretation of primitive atoms in \mathbf{L} . This obtained expression is finite for any finite formula (or abstracted term), and represents the *meaning* of such finite formula (or abstracted term).

The *extension* of abstracted terms satisfy the following property:

Proposition 3 *For any abstracted term $\langle \phi \rangle_{\alpha}^{\beta}$ with $|\alpha| \geq 1$ we have that $h(I(\langle \phi \rangle_{\alpha}^{\beta})) = \pi_{-\beta}(h(I(\phi)))$, where $\pi_{-(y_1, \dots, y_k)} = \pi_{-y_1} \circ \dots \circ \pi_{-y_k}$, \circ is the sequential composition of functions, and $\pi_{-\emptyset}$ is an identity.*

Proof: Let \mathbf{x} be a tuple of all free variables in ϕ , so that $\bar{\mathbf{x}} = \bar{\alpha} \cup \bar{\beta}$, $\alpha = (x_1, \dots, x_k)$, then we have that $h(I(\langle \phi \rangle_{\alpha}^{\beta})) = h(I(\text{union}(\{I(\phi[\beta/g(\beta)]) \mid g \in \mathcal{D}^{\bar{\beta}}\})))$, from Def. 8
 $= \bigcup \{h(I(\phi[\beta/g(\beta)])) \mid g \in \mathcal{D}^{\bar{\beta}}\}$
 $= \bigcup \{(g_1(x_1), \dots, g_1(x_k)) \mid g_1 \in \mathcal{D}^{\bar{\alpha}} \text{ and } h(I(\phi[\beta/g(\beta)])) = t\} \mid g \in \mathcal{D}^{\bar{\beta}}\}$
 $= \{g_1(\alpha) \mid g_1 \in \mathcal{D}^{\bar{\alpha} \cup \bar{\beta}} \text{ and } h(I(\phi/g_1)) = t\}$
 $= \pi_{-\beta}(\{g_1(\mathbf{x}) \mid g_1 \in \mathcal{D}^{\bar{\mathbf{x}}} \text{ and } h(I(\phi/g_1)) = t\})$
 $= \pi_{-\beta}(\{g_1(\mathbf{x}) \mid g_1 \in \mathcal{D}^{\bar{\mathbf{x}}} \text{ and } g_1(\mathbf{x}) \in h(I(\phi))\})$, by (T)
 $= \pi_{-\beta}(h(I(\phi)))$.

\square

We can connect \mathcal{E} with a possible-world semantics. Such a correspondence is a natural identification of intensional logics with modal Kripke based logics.

Definition 9 (Model): *A model for intensional FOL with fixed intensional interpretation I , which express the two-step intensional semantics in Definition 6, is the Kripke structure $\mathcal{M}_{int} = (\mathcal{W}, \mathcal{D}, V)$, where $\mathcal{W} = \{is^{-1}(h) \mid h \in \mathcal{E}\}$, a mapping $V : \mathcal{W} \times P \rightarrow \bigcup_{n < \omega} \{t, f\}^{\mathcal{D}^n}$, with P a set of predicate symbols of the language, such that for any world $w = is^{-1}(h) \in \mathcal{W}$, $p_i^n \in P$, and $(u_1, \dots, u_n) \in \mathcal{D}^n$ it holds that $V(w, p_i^n)(u_1, \dots, u_n) = h(I(p_i^n(u_1, \dots, u_n)))$. The satisfaction relation $\models_{w,g}$ for a given $w \in \mathcal{W}$ and assignment $g \in \mathcal{D}\mathcal{V}$ is defined as follows:*

1. $\mathcal{M} \models_{w,g} p_i^k(x_1, \dots, x_k)$ iff $V(w, p_i^k)(g(x_1), \dots, g(x_k)) = t$,
2. $\mathcal{M} \models_{w,g} \varphi \wedge \phi$ iff $\mathcal{M} \models_{w,g} \varphi$ and $\mathcal{M} \models_{w,g} \phi$,
3. $\mathcal{M} \models_{w,g} \neg \varphi$ iff not $\mathcal{M} \models_{w,g} \varphi$,
4. $\mathcal{M} \models_{w,g} (\exists x) \phi$ iff
 - 4.1. $\mathcal{M} \models_{w,g} \phi$, if x is not a free variable in ϕ ;
 - 4.2. exists $u \in \mathcal{D}$ such that $\mathcal{M} \models_{w,g} \phi[x/u]$, if x is a free variable in ϕ .

It is easy to show that the satisfaction relation \models for this Kripke semantics in a world $w = is^{-1}(h)$ is defined by, $\mathcal{M} \models_{w,g} \phi$ iff $h(I(\phi/g)) = t$.

We can enrich this intensional FOL by another modal operators, as for example the "necessity" universal operator \Box with an accessibility relation $\mathcal{R} = \mathcal{W} \times \mathcal{W}$, obtaining the S5 Kripke structure $\mathcal{M}_{int} = (\mathcal{W}, \mathcal{R}, \mathcal{D}, V)$, in order to be able to define the following equivalences between the abstracted terms without free variables $\langle \phi \rangle_{\alpha}^{\beta_1} / g$ and $\langle \psi \rangle_{\alpha}^{\beta_2} / g$, where all free variables (not in α) are instantiated by $g \in \mathcal{D}^{\mathcal{V}}$ (here $A \equiv B$ denotes the formula $(A \Rightarrow B) \wedge (B \Rightarrow A)$):

- (Strong) intensional equivalence (or *equality*) " \asymp " is defined by:
 $\langle \phi \rangle_{\alpha}^{\beta_1} / g \asymp \langle \psi \rangle_{\alpha}^{\beta_2} / g$ iff $\Box(\phi[\beta_1/g(\beta_1)] \equiv \psi[\beta_2/g(\beta_2)])$,
 with $\mathcal{M} \models_{w,g'} \Box \varphi$ iff for all $w' \in \mathcal{W}$, $(w, w') \in \mathcal{R}$ implies $\mathcal{M} \models_{w',g'} \varphi$.
 From Example 1 we have that $\langle p_1^1(x) \rangle_x \asymp \langle p_2^1(x) \rangle_x$, that is 'x has been bought' and 'x has been sold' are intensionally equivalent, but they have not the same meaning (the concept $I(p_1^1(x)) \in D_1$ is different from $I(p_2^1(x)) \in D_1$).
- Weak intensional equivalence " \approx " is defined by:
 $\langle \phi \rangle_{\alpha}^{\beta_1} / g \approx \langle \psi \rangle_{\alpha}^{\beta_2} / g$ iff $\Diamond \phi[\beta_1/g(\beta_1)] \equiv \Diamond \psi[\beta_2/g(\beta_2)]$.
 The symbol $\Diamond = \neg \Box \neg$ is the correspondent existential modal operator.

This weak equivalence is used for P2P database integration in a number of papers [33, 34, 35, 36, 37, 38, 31].

Notice that we do not use the intensional equality in our language, thus we do not need the correspondent operator in intensional algebra Alg_{int} for the logic "necessity" modal operator \Box .

This semantics is equivalent to the algebraic semantics for L in [25] for the case of the conception where intensional entities are considered to be *equal* if and only if they are *necessarily equivalent*. Intensional equality is much stronger than the standard *extensional equality* in the actual world, just because requires the extensional equality in *all* possible worlds, in fact, if $\langle \phi \rangle_{\alpha}^{\beta_1} / g \prec \langle \psi \rangle_{\alpha}^{\beta_1} / g$ then $h(I(\langle \phi \rangle_{\alpha}^{\beta_1} / g)) = h(I(\langle \psi \rangle_{\alpha}^{\beta_2} / g))$ for all extensionalization functions $h \in \mathcal{E}$ (that is possible worlds $is^{-1}(h) \in \mathcal{W}$).

It is easy to verify that the intensional equality means that in every possible world $w \in \mathcal{W}$ the intensional entities u_1 and u_2 have the same extensions.

Let the logic modal formula $\Box\phi[\beta_1/g(\beta_1)]$, where the assignment g is applied only to free variables in β_1 of a formula ϕ not in the list of variables in $\alpha = (x_1, \dots, x_n)$, $n \geq 1$, represents a n-ary intensional concept such that $I(\Box\phi[\beta_1/g(\beta_1)]) \in D_n$ and $I(\phi[\beta_1/g(\beta_1)]) = I(\langle \phi \rangle_{\alpha}^{\beta_1} / g) \in D_n$. Then the extension of this n-ary concept is equal to (here the mapping $necess : D_i \rightarrow D_i$ for each $i \geq 0$ is a new operation of the intensional algebra Alg_{int} in Definition 7):

$$\begin{aligned} h(I(\Box\phi[\beta_1/g(\beta_1)])) &= h(necess(I(\phi[\beta_1/g(\beta_1)]))) = \\ &= \{(g'(x_1), \dots, g'(x_n)) \mid \mathcal{M} \models_{w, g'} \Box\phi[\beta_1/g(\beta_1)] \text{ and } g' \in \mathcal{D}^{\mathcal{V}}\} \\ &= \{(g'(x_1), \dots, g'(x_n)) \mid g' \in \mathcal{D}^{\mathcal{V}} \text{ and } \forall w_1 ((w, w_1) \in \mathcal{R} \text{ implies } \mathcal{M} \models_{w_1, g'} \phi[\beta_1/g(\beta_1)])\} \\ &= \bigcap_{h_1 \in \mathcal{E}} h_1(I(\phi[\beta_1/g(\beta_1)])). \end{aligned}$$

While,

$$\begin{aligned} h(I(\Diamond\phi[\beta_1/g(\beta_1)])) &= h(I(\neg\Box\neg\phi[\beta_1/g(\beta_1)])) \\ &= h(neg(necess(I(\neg\phi[\beta_1/g(\beta_1)])))) \\ &= \mathcal{D}^n \setminus h(necess(I(\neg\phi[\beta_1/g(\beta_1)]))) \\ &= \mathcal{D}^n \setminus (\bigcap_{h_1 \in \mathcal{E}} h_1(I(\neg\phi[\beta_1/g(\beta_1)]))) \\ &= \mathcal{D}^n \setminus (\bigcap_{h_1 \in \mathcal{E}} h_1(neg(I(\phi[\beta_1/g(\beta_1)])))) \\ &= \mathcal{D}^n \setminus (\bigcap_{h_1 \in \mathcal{E}} \mathcal{D}^n \setminus h_1(I(\phi[\beta_1/g(\beta_1)]))) \\ &= \bigcup_{h_1 \in \mathcal{E}} h_1(I(\phi[\beta_1/g(\beta_1)])). \end{aligned}$$

Consequently, the concepts $\Box\phi[\beta_1/g(\beta_1)]$ and $\Diamond\phi[\beta_1/g(\beta_1)]$ are the *built-in* (or rigid) concept as well, whose extensions does not depend on possible worlds.

Thus, two concepts are intensionally *equal*, that is, $\langle \phi \rangle_{\alpha}^{\beta_1} / g \prec \langle \psi \rangle_{\alpha}^{\beta_2} / g$, iff $h(I(\phi[\beta_1/g(\beta_1)])) = h(I(\psi[\beta_2/g(\beta_2)]))$ for every h . Moreover, two concepts are *weakly equivalent*, that is, $\langle \phi \rangle_{\alpha}^{\beta_1} / g \approx \langle \psi \rangle_{\alpha}^{\beta_2} / g$,

$$\text{iff } h(I(\Diamond\phi[\beta_1/g(\beta_1)])) = h(I(\Diamond\psi[\beta_2/g(\beta_2)])).$$